

Heisenberg constraints on mesoscopic and molecular amplifiers

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Heisenberg uncertainty relations for current components impose constraints on the performance of linear amplifiers. Here we derive such constraints for amplifiers in which the input signal modulates a bias current in order to produce an amplified output. These amplifiers include transistors, macroscopic, mesoscopic, or molecular, operated as linear amplifiers.

1 Introduction

Commutation relations or Heisenberg uncertainty relations for observable associated with the inputs and outputs of amplifiers have played an important role in determining the optimum performance that can be achieved by amplifiers and detectors^{1–14}. Most of these discussions have focused on maser, laser, and optical parametric amplifiers, the first devices to achieve nearly quantum limited performance. The arguments that establish the quantum limited performance of amplifiers of the electromagnetic field, such as optical amplifiers, do not directly carry over to devices that employ fermionic currents. It is thus worth addressing the issue of the quantum limits of amplifier performance in a way that is directly applicable to semiconductor devices, particularly since semiconductor device development, such as in the case of single electron transistors^{8,12}, has proceeded to the point where quantum limited performance seems to be within reach. Here we present further results in our investigation¹⁶ of quantum mechanical restrictions on transistor amplifier performance.

Let I_{in} denote the current delivered by a signal source to the input of an amplifier and I_{out} denote the current delivered by the amplifier to a load. Ideally, the relation between these two currents would be

$$I_{out}(t) = G_p \sqrt{\frac{g_l}{g_s}} I_{in}(t), \quad (1)$$

where G_p^2 is the power gain and g_s and g_l are the *differential conductances* of the source and load respectively. However, for $G_p \neq 1$, the current-current commutation relations required by quantum mechanics cannot be satisfied. This situation is remedied by replacing Eq. (2) with

$$I_{out}(t) = G_p \sqrt{\frac{g_l}{g_s}} I_{in}(t) + I_N(t), \quad (2)$$

where $I_N(t)$ is a current operator associated with noise generated within the amplifier. As noise, I_N is independent of I_{in} and the two commute: $[I_{in}, I_N] = 0$.

We consider the case when the current $I(t)$ is investigated with detectors that respond over only over a frequency window $\Delta\omega$ around a center frequency ω_0 . Introducing the fourier transform

$$I(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega I(\omega) e^{-i\omega t}, \quad (3)$$

the current sensed by the detectors is then given by

$$I(t) = \frac{1}{\sqrt{2\pi}} \bar{I}(\omega_0) + H.c \quad (4)$$

where we defined the band-integrated current transform (analogous to the annihilation operator of the harmonic oscillator) by

$$\bar{I}(\omega_0) \equiv \int_{\omega_0 \pm \frac{1}{2}\Delta\omega} I(\omega) e^{-i\omega t} d\omega. \quad (5)$$

For an ideal power amplification (i.e. when one is interested in transferring maximum power from the system to the amplifier and from the amplifier to a load resistor) in a stationary (though nonequilibrium) state it has been shown¹⁵ in Ref. 16 that I_N satisfies

$$\langle [\bar{I}_N(\omega_0), \bar{I}_N^\dagger(\omega_0)] \rangle = -(G_p^2 - 1) \frac{\hbar\omega_0}{2} g_\ell \Delta\omega \quad (6)$$

and

$$\Delta I_N^2(t) \geq (G_p^2 - 1) \frac{\hbar\omega_0}{2} g_\ell \Delta\nu \quad (7)$$

provided $\Delta\nu = \Delta\omega/2\pi \ll \omega_0$, where $\Delta A \equiv (\langle A^2 \rangle - \langle A \rangle^2)^{1/2}$. Eq.7 is a very general constraint on the minimal noise added in linear amplification. The basic assumptions in its derivation are that the amplifier is linear (i.e. that Eq.2 holds), that the total system is in a stationary state, and that the differential conductance of the amplifier remains constant independently of the input current signal. These assumptions make possible the derivation through the application of Kubo's fluctuation-dissipation theorem^{17,18} generalized to nonequilibrium steady states^{19–22}.

We turn now to considering a more specific class of devices, namely transistor amplifiers (such as, for example, field effect, single-electron, or molecular transistors). A typical feature of these devices is that they operate in a nonequilibrium current carrying state even in the absence of coupling to an input signal. This nonequilibrium current is accompanied by *shot-noise* - the nonequilibrium current fluctuations. It is therefore natural to ask whether the constraint Eq.7 can be refined to take the existence of this noise into account. The positive answer to this question is stated in the next section, and derived in the last one.

2 Main result

Consider a specific case of a linear amplifier operating in a stationary state where current is flowing through it even in the absence of a coupling to any signal. In this case it is useful to write I_N in Eq.2 as a sum of two currents:

$$I_N = I_0 + I_n \quad (8)$$

where I_0 is the current of the amplifier before the coupling interaction between the signal and the amplifier is turned on and I_n is the change in I_N due to switching on the coupling. Assume now that the coupling is proportional to a small dimensionless parameter, γ . Since I_0 existed before γ was switched on, it is of zeroth order in γ . I_n appeared as a result of the coupling and therefore it is of higher order in γ . Also the power gain G_p is of higher order in γ since no coupling implies no gain. We assume that I_n is of higher order in γ than the gain. Our main result states that the following inequality must be satisfied:

$$\Delta I_0(t) \Delta I_n(t) \geq \frac{1}{4} G_p^2 \hbar\omega_0 g_\ell \Delta\nu. \quad (9)$$

Eq.9 has several nontrivial consequences. For example, it implies that the "old" shot-noise $\Delta I_0^2(t)$ is *necessary* for an ideal operation of the amplifier since coupling a device with vanishing shot-noise to a signal will result in the appearance of "new" shot noise $\Delta I_n^2(t)$ which should diverge in order to maintain the inequality in Eq.9.

3 Derivation of the Heisenberg constraint

To derive Eq.9 we make use of Eq.6 twice, first in the presence and then in the absence of the coupling γ : The current noise in these cases is given by

$$\begin{aligned} I_N &= I_0 + I_n & \gamma &\neq 0 \\ I_N &= I_0 & \gamma &= 0 \end{aligned} \quad (10)$$

Inserting these into Eq.6 yields

$$\begin{aligned} \langle [\bar{I}_0(\omega_0), \bar{I}_0^\dagger(\omega_0)] \rangle + \langle [\bar{I}_n(\omega_0), \bar{I}_0^\dagger(\omega_0)] \rangle + \langle [\bar{I}_0(\omega_0), \bar{I}_n^\dagger(\omega_0)] \rangle + \langle [\bar{I}_n(\omega_0), \bar{I}_n^\dagger(\omega_0)] \rangle \\ = -(G_p^2 - 1) \frac{\hbar\omega_0}{2} g_\ell \Delta\omega \end{aligned} \quad (11)$$

and

$$\langle [\bar{I}_0(\omega_0), \bar{I}_0^\dagger(\omega_0)] \rangle = \frac{\hbar\omega_0}{2} g_\ell \Delta\omega \quad (12)$$

where we have used the fact that $G = 0$ when $\gamma = 0$. Subtracting the last equation from the previous one, one gets

$$\langle [\bar{I}_n(\omega_0), \bar{I}_0^\dagger(\omega_0)] \rangle + \langle [\bar{I}_0(\omega_0), \bar{I}_n^\dagger(\omega_0)] \rangle + \langle [\bar{I}_n(\omega_0), \bar{I}_n^\dagger(\omega_0)] \rangle = -G_p^2 \frac{\hbar\omega_0}{2} g_\ell \Delta\omega. \quad (13)$$

Since we assumed that I_n is of higher order in γ than the gain, the term $\langle [\bar{I}_n(\omega_0), \bar{I}_n^\dagger(\omega_0)] \rangle$ is of higher order in γ than the three other terms in Eq.13. Since this equation should hold for any value of γ small enough for the amplifier to be regarded as linear, $\langle [\bar{I}_n(\omega_0), \bar{I}_n^\dagger(\omega_0)] \rangle$ must vanish. Thus, Eq.13 becomes:

$$\langle [\bar{I}_n(\omega_0), \bar{I}_0^\dagger(\omega_0)] \rangle + \langle [\bar{I}_0(\omega_0), \bar{I}_n^\dagger(\omega_0)] \rangle = -G_p^2 \frac{\hbar\omega_0}{2} g_\ell \Delta\omega. \quad (14)$$

We now use the fact that for any pair of hermitian Heisenberg operators $A_1(t)$ and $A_2(t)$, one has

$$\langle \bar{A}_i(\omega_0) \rangle = 0 \quad \omega_0 \neq 0 \quad i = 1, 2, \quad (15)$$

and

$$\langle \bar{A}_2(\omega_0) \bar{A}_1(\omega_0) \rangle = \langle \bar{A}_1(\omega_0) \bar{A}_2(\omega_0) \rangle = 0 \quad (16)$$

where $\bar{A}_i(\omega_0) = \int_{\omega_0 \pm \frac{1}{2}\Delta\omega} d\omega \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt_i e^{i\omega t} A_i(t_i)$, $i = 1, 2$, provided that the averages are performed in a stationary state (the proof of Eq.16 is straightforward by substitution of the definition of $\bar{A}_i(\omega_0)$, making a change of the integration variables $\tau_1 = t_1 - t_2$, $\tau_2 = \frac{1}{2}(t_1 + t_2)$ and integrating over τ_2). Taking $A_1 = I_0$ and $A_2 = I_n$, Eq.16 enables us to rewrite Eq.14 in the form of an expectation value of a commutator of two hermitian operators $\bar{I}_n(\omega_0) + \bar{I}_n^\dagger(\omega_0)$ and $i(\bar{I}_0^\dagger(\omega_0) - \bar{I}_0(\omega_0))$ which are analogous to a position and a momentum operator, respectively, or to the field quadrature components of quantum optics:

$$\langle [\bar{I}_n(\omega_0) + \bar{I}_n^\dagger(\omega_0), i(\bar{I}_0^\dagger(\omega_0) - \bar{I}_0(\omega_0))] \rangle = -iG_p^2 \frac{\hbar\omega_0}{2} g_\ell \Delta\omega. \quad (17)$$

This implies the uncertainty relation

$$\Delta(\bar{I}_n(\omega_0) + \bar{I}_n^\dagger(\omega_0)) \Delta(i(\bar{I}_0^\dagger(\omega_0) - \bar{I}_0(\omega_0))) \geq \frac{1}{2} G_p^2 \frac{\hbar\omega_0}{2} g_\ell \Delta\omega. \quad (18)$$

Eqs. 15 and 16 also imply (together with Eq.4):

$$\begin{aligned}\Delta(\bar{I}_n(\omega_0) + \bar{I}_n^\dagger(\omega_0))^2 &= 2\pi\langle I_n^2(t)\rangle \\ \Delta(i(\bar{I}_0^\dagger(\omega_0) - \bar{I}_0(\omega_0)))^2 &= 2\pi\langle I_0^2(t)\rangle.\end{aligned}\tag{19}$$

Finally, substituting the last two equalities into Eq.18 one recovers the constraint, Eq.9.

To conclude, a novel Heisenberg constraint on shot-noise carrying linear amplifier was obtained. This constraint relates the device shot noise before coupling to the signal and the one added due to this coupling. One consequence of this relation is that an attempt to indefinitely reduce the shot-noise in the device in the absence of a signal will result in the appearance of diverging new shot-noise after the coupling to the signal is switched on.

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